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THE DIFFRACTION OF A HIGH-FREQUENCY ACOUSTIC WAVE BY A NARROW-ANGLE ABSOLUTELY RIGID CONE OF ARBITRARY SHAPE†

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When a plane acoustic wave or a wave generated by a point source of oscillations is diffracted by an absolutely rigid cone of arbitrary shape, a wave with a spherical wavefront is produced which spreads out from the vertex of the cone as from the centre. For a narrow-angle cone, with certain limitations on the directions in which observations are made, explicit approximate formulae are constructed for the radiation patterns of this wave, which generalizes the formulae obtained in [1, 2] for a circular cone.

1. FORMULATION OF THE PROBLEM

Suppose an absolutely rigid cone Ξ has a vertex C, which coincides with the origin of coordinates. We will assume that the wave process is described by Helmholtz' equation $(\Delta + k^2)\,u = 0$, where u is the velocity potential, and a wave u_i propagating either from a point source situated at the point $\mathbf{r}_0 = (x_1^0, x_2^0, x_3^0)$ is incident on the cone, in which case

$$u_i = (4\pi R)^{-1}\exp(ikR), \quad R = |\mathbf{r} - \mathbf{r}_0|, \quad \mathbf{r} = (x_1, x_2, x_3) \tag{1.1}$$

or a plane wave, in which case

$$u_i = \exp(-ik(\omega_0, r)), \quad |\omega_0| = 1 \tag{1.2}$$

The unit vector ω_0 indicates the direction in which the plane wave propagates. We will assume that in the first case, when formula (1.1) holds, the vector $\mathbf{r}_0$ is also parallel to the vector ω_0, or more correctly

$$\mathbf{r}_0 = \omega_0 r_0, \quad \mathbf{r}_0 = |r_0|$$

Suppose u_s is the wave scattered by the cone. Then the velocity potential $u = u_i + u_s$ must satisfy the no-flow condition

$$\left.\frac{\partial u}{\partial \zeta}\right|_{\partial\Xi} = 0 \tag{1.3}$$

where $\partial\Xi$ is the surface of the cone Ξ, and $\partial/\partial\zeta$ represents differentiation along the normal, Helmholtz's equation, Meixner's condition of the finiteness of the sound energy in the neighbourhood of C, and the corresponding form of the radiation conditions.

As we know, u_i generates, among other waves, a spherical wave G_{diff}, which spreads out from the vertex of the cone C as from the centre. We have the following formulae for G_{diff}

$$G_{\text{diff}}^{(1)} = (2krr_0)^{-1}\exp(ik(r + r_0))\,f(\omega, \omega_0) + O((kr)^{-2}) \tag{1.4}$$
$$G_{\text{diff}}^{(2)} = 2\pi(kr)^{-1}\exp(ikr)\,f(\omega, \omega_0) + O((kr)^{-2})$$

in cases (1.1) and (1.2) respectively. Here $\omega(|\,\omega\,| = 1)$ is the unit vector in the direction in which observations are made (i.e. the radius vector of the point of observation $\mathbf{r} = r\omega$, $kr \gg 1$). Knowing the

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radiation pattern $f(\omega, \omega_0)$ we can obtain $G^{(j)}_{\text{diff}}$ ($j = 1, 2$) in the first approximation. The purpose of this paper is to derive an approximate formula for $f(\omega, \omega_0)$ in the case of a narrow-angle cone.

2. SMYSHLYAYEV'S FORMULA FOR $f(\omega, \omega_0)$

Smyshlyayev [3]† has derived the following general formula

$$f(\omega,\omega_0) = \frac{i}{\pi}\int_\gamma e^{-i\nu\pi} g(\omega,\omega_0,\nu)\,\nu\,d\nu \tag{2.1}$$

The notation used here, which is similar to that used in [3], needs to be explained. Suppose S^2 is the unit sphere with centre at C. Suppose N is the part of S^2 which does not belong to the cone Ξ, i.e. $N = S^2\backslash\Xi$ (see Fig. 1) and g is Green's function of the Helmholtz operator for the region M, or more accurately g is the solution of the problem

$$\left(\Delta_S + \nu^2 - \frac{1}{4}\right) g = \delta(\omega - \omega_0), \quad \left.\frac{\partial g}{\partial m}\right|_{\partial N} = 0 \tag{2.2}$$

$$\Delta_S = \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

The differentiation is carried out with respect to the coordinates of the point ω, ∂N is the boundary of the region $N \subset S^2$, $\omega_0 \in N$ is a fixed point, and $\partial/\partial m$ means differentiation along the sphere along the normal to ∂N.

The integration in (2.1) is carried out along the contour γ in the complex plane ν enveloping all the poles of the function g. The function $f(\omega, \omega_0)$ has singularities corresponding to breakdowns in the regularity of the field of the rays of geometrical optics. This is the case, for example, for points which satisfy the condition

$$\min_{s\in\partial N}(\text{dist}(\omega,s) + \text{dist}(s,\omega_0)) = \pi \tag{2.3}$$

(dist (a, b) is the distance between a, b ($a, b \in S^2$) along the sphere S^2). The direction along which the wavefront reflected from the surface of the cone touches the wavefront scattered by the vertex of the cone satisfies condition (2.3).

We will consider the following case, which is important in practice, when instead of (2.3) we have the inequality

$$\min_{s\in\partial N}[\text{dist}(\omega,s) + \text{dist}(s,\omega_0)] > \pi \tag{2.4}$$

In this case (2.1) can be replaced [3] by

$$f(\omega,\omega_0) = -i/\pi\int_{-\infty}^{+\infty} e^{\pi\tau} g_r(\omega,\omega_0)\,\tau\,d\tau \tag{2.5}$$

$$g = g_0 + g_r = -\frac{1}{4\,\text{ch}\,\pi\tau} P_{-\frac{1}{2}+i\tau}(-\cos\hat{\theta}) + g_r \tag{2.6}$$

$$\left.\frac{\partial}{\partial m} g_r\right|_{\partial N} = -\left.\frac{\partial}{\partial m} g_0\right|_{\partial N}, \quad \left(\Delta_s - \left(\tau^2 + \frac{1}{4}\right)\right) g_r = 0 \tag{2.7}$$

Here $g_r(\omega, \omega_0)$ is the so-called reflected part of Green's function, P_ν is Legendre's function, g_0 is Green's function of the operator $\Delta_s - (\tau^2 + 1/4)$ for the case when $\partial N = \varnothing$ and $N = S^2$, i.e. g_0 is Green's function for the whole sphere.

Note that another approach to the problem of the diffraction of waves by an arbitrary cone was proposed in [4].

†See also: SMYSHLYAYEV V. P. On the diffraction of waves by cones at high frequencies. Leningrad, Optiko-Mekh. Inst., Preprint E-9-89, Leningrad, 1989.

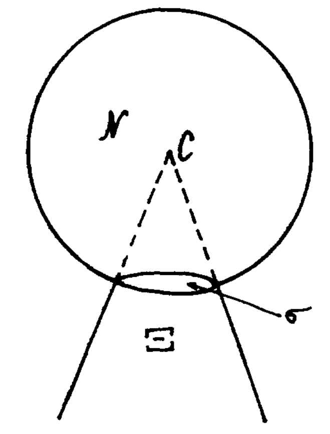

Fig. 1.

3. THE OUTER AND INNER REGIONS

If the diameter of the region cut off by the cone Ξ from S^2 (i.e. the region $S^2\backslash\bar{N}$), is small, the cone Ξ will be said to be narrow-angled. Suppose this diameter is of the order of ε, where ε is a small parameter, and hence the region $S^2\backslash\bar{N}$ itself will be denoted by σ_ε. We will assume that the point $O \in \sigma_\varepsilon$. Drawing a tangential plane $\mathbf{R}^2$ at the point O and projecting σ_ε onto $\mathbf{R}^2$, we obtain a certain region $\kappa_\varepsilon \subset \mathbf{R}^2$. Suppose x_1, x_2, x_3 is a coordinate system in $\mathbf{R}^3$ with origin at the point O, while the x_3 axis is directed into the sphere. The equation of the sphere will obviously have the form $x_3 = 1 - \sqrt{(1 - x_1^2 - x_2^2)}$ and in the region of the point O the variables x_1, x_2 can be regarded as coordinates on the sphere.

We will assume that the boundary $\partial\kappa_\varepsilon$ of the region κ_ε is "small", or, more exactly, that the points $x_1, x_2 \in \partial\kappa_\varepsilon$ are obtained by a similar transformation from the points X_1, X_2 of the fixed close curve $\partial\kappa$

$$\partial\kappa_\varepsilon = \{(x_1, x_2): x_i = \varepsilon X_i,\ \ i = 1, 2,\ \ (X_1, X_2) \in \partial\kappa\},\ \ \varepsilon > 0,\ \ \varepsilon \ll 1$$

i.e. $\partial\kappa_\varepsilon$ is the result of a similar transformation of the fixed curve $\partial\kappa$ with similitude coefficient ε.

We will further obtain, using the well-known method described in [5], the leading term of the asymptotic form of g_r as $\varepsilon \to 0$. Substituting it into (2.5) we obtain an approximate expression for $f(\omega, \omega_0)$ in terms of elementary functions. To obtain the asymptotic form of g_r a correct choice of the form of the asymptotic expansion in the region $\partial\kappa_\varepsilon$ (the inner region) and outside a small neighbourhood of $\partial\kappa_\varepsilon$ (the outer region) is of fundamental importance.

Suppose E_1, E_2, α, β $(0 < \alpha < \beta < 1)$ are certain positive constants. For sufficiently small ε we have $E_1\varepsilon^\alpha > E_2\varepsilon^\beta$.

We will refine the idea of the outer region: this is the region that is obtained if we remove from N all points which project into the circle

$$r = \sqrt{x_1^2 + x_2^2} \leqslant E_2\varepsilon^\beta \tag{3.1}$$

In view of the fact that ε is a small parameter, while the coordinates of the curve $\partial\kappa_\varepsilon$ are of the order of ε, the curve $\partial\kappa_\varepsilon$ lies inside the circle (3.1).

The inner region is the region on the sphere which is obtained if we remove from the points of the sphere which project into the open circle

$$r = \sqrt{x_1^2 + x_2^2} < E_1\varepsilon^\alpha \tag{3.2}$$

the points of the closed region $\bar{\kappa}_\varepsilon$.

The equation and boundary conditions in the inner region are best considered in extended coordinates, for which we must write both the equation and the boundary condition (2.7) in x_1, x_2 coordinates.

We will begin from the equation. Considering x_1, x_2 as coordinates on the sphere S^2, we initially obtain the square of the differential of the length of an arbitrary curve on S^2

$$\sum_{h=1}^{3} (dx_h)^2 = \sum_{i=1}^{2} (dx_i)^2 + \left(\sum_{j=1}^{2} \frac{\partial}{\partial x_j} dx_j \right)^2 := a_{ij} dx_i dx_j \tag{3.3}$$

$$a_{ij} = \frac{x_i x_j}{1 - x_1^2 - x_2^2} + \delta_{ij} \tag{3.4}$$

Here, and also henceforth, summation over repeated indices is assumed. The operator Δ_s (see (2.7)) has the following form in x_1, x_2 coordinates

$$\Delta_S = \frac{1}{\sqrt{a}} \frac{\partial}{\partial x_i} \left(a^{ij} \sqrt{a} \frac{\partial}{\partial x_j} \right), \qquad a = \det \| a_{ij} \| \tag{3.5}$$

where $\| \alpha^{ij} \|$ is the inverse matrix to $\| \alpha_{ij} \|$.

Equation (2.7) can be written as follows in extended coordinates $X_i = x_i/\varepsilon$ $(i = 1, 2)$

$$\left[\frac{\varepsilon^2}{\sqrt{A}} \frac{\partial}{\partial X_i} \left(\sqrt{A}\, A^{ij} \frac{\partial}{\partial X_j} \right) - \left(\tau^2 + \frac{1}{4} \right) \right] g_r = 0 \tag{3.6}$$

$$A = \det(I + O(\varepsilon^2)) = 1 + O(\varepsilon^2), \qquad \sqrt{A}\, A^{ij} = \delta_{ij} + O(\varepsilon^2)$$

Here I is the unit matrix and $O(\varepsilon^2)$ are different algebraic expressions of X_i and ε which, when $X_i = 0(1)$ and $\varepsilon \to 0$ are of order no less than ε^2.

The boundary condition on the boundary $\partial\kappa_\varepsilon$ in x_i coordinates has the form

$$\frac{\partial g_r}{\partial x_i} \cos \hat{n x_j} a^{ij} = - \frac{\partial g_0}{\partial x_i} \cos \hat{n x_j} a^{ij} \tag{3.7}$$

Here $\cos \hat{n x_j}$ $(j = 1, 2)$ are the direction cosines of the usual Euclidean normal to the curve $\partial\kappa_\varepsilon$ in the $\mathbf{R}^2$ plane.

Introducing extended coordinates X_i, we obtain that boundary condition (3.7) must be satisfied on the curve $\partial\kappa$, which does not depend on the small parameter ε. This will have the form

$$A^{ij} \frac{\partial g_r}{\partial X_i} \cos \hat{n X_j} = - A^{ij} \frac{\partial g_0}{\partial X_i} \cos \hat{n X_j} \tag{3.8}$$

4. THE OUTER AND INNER EXPANSIONS

In the outer region we will seek an expansion of g_r in the form of a series in multipoles. Suppose g_0 is the fundamental solution corresponding to the operator $\Delta_S - (\tau^2 + 1/4)$

$$g_0 = g_0(\omega, \omega_2) = -(4 \operatorname{ch} \pi\tau)^{-1} P_{-\frac{1}{2}+i\tau}(-\cos\theta_1), \quad \theta_1 = \operatorname{dist}(\omega, \omega_1)$$

where $\omega_1 = \omega_1(x_1^0, x_2^0)$ belongs to the circle (3.2).

The outer expansion will be sought in the form

$$g_r = \varepsilon B_1 g_0 + \varepsilon^2 \left(B_2 g_0 + B_{2j} \frac{\partial g_0}{\partial x_j^0} \right) + \varepsilon^3 \left(B_3 g_0 + B_{3j} \frac{\partial g_0}{\partial x_j^0} + B_{3ij} \frac{\partial^2 g_0}{\partial x_i^0 \partial x_j^0} \right) + \ldots \tag{4.1}$$

The arguments of the function g_0 and its derivatives are the points ω, $O \in S^2$, where $B_1, B_2, B_{2j}, \ldots$ are coefficients to be determined. The functions $\partial g_0/\partial x_j^0$, $\partial^2 g_0/\partial x_i^0\, \partial x_j^0$ (differentiation is carried out with respect to the coordinates of the point O) are naturally called multipoles.

For the inner expansion we put

$$g_r = \sum_{j=1}^{\infty} U_j(X_1, X_2)\,\varepsilon^j, \quad X_i = \frac{x_i}{\varepsilon} \tag{4.2}$$

and we match the asymptotic expansions (4.1) and (4.2) in the region (see Section 3)

$$E_2\varepsilon^\beta < r < E_1\varepsilon^\alpha, \quad E_j = \text{const}, \quad 0 < \alpha < \beta < 1 \tag{4.3}$$

We will now construct the terms of the series (4.1) and (4.2) using the well-known method described in [5].

We formally substitute series (4.2) into Eq. (2.7), assuming that the operator Δ_s is written in X_i variables (see (3.6)). Equating terms of the order of $1/\varepsilon$ to zero, we obtain

$$\Delta U_1 = 0, \quad \Delta = \partial^2 / \partial X_1^2 + \partial^2 / \partial X_2^2 \tag{4.4}$$

In region (4.3) $R \equiv \sqrt{(X_1^2 + X_2^2)} \to \infty$, while $\sqrt{(x_1^2 + x_2^2)} \to 0$ as $\varepsilon \to 0$. Taking into account the fact that expansions (4.1) and (4.2) in region (4.3) must be asymptotically equivalent and that $g_0 \sim (2\pi)^{-1} \ln r$ as $\omega \to 0$, we obtain, as $R \to \infty$

$$U_1 \sim (2\pi)^{-1} B_1 \ln R\varepsilon + O(R^{-1}) \tag{4.5}$$

Boundary condition (2.7) can be written in X_i coordinates as follows:

$$\left.\frac{1}{\varepsilon} A^{ij} \frac{\partial g_r}{\partial X_j} \cos \hat{n}\, x_i \right|_{\partial\kappa} = \left. -\frac{1}{\varepsilon} A^{ij} \frac{\partial g_0}{\partial X_j} \cos \hat{n}\, x_i \right|_{\partial\kappa} \tag{4.6}$$

where n is the normal to the curve $\partial\kappa$. To fix our ideas we will assume that n is the outward normal to the finite region $\kappa \subset \mathbf{R}^2$, bounded by the curve $\partial\kappa$. We replace the function g_0 by its expansion in a Taylor series in the neighbourhood of the point O

$$g_0 = D_0 + D_i x_i + 1/2 D_{ij} x_i x_j + \ldots, \quad D_0 = g_0(\omega_0, O), \quad D_i = (\partial g_0 / \partial x_i)_O \ldots \tag{4.7}$$

Taking relations (4.2), (4.6) and (3.7) into account we obtain

$$\partial U_1 / \partial n|_{\partial\kappa} = -D_i \partial X_i / \partial n \tag{4.8}$$

i.e. U_1 is the solution of the Neumann problem (4.4), (4.5) and (4.8).

Using the fact that the integral of the normal derivative of an harmonic function over a closed curve does not change when this curve is deformed, we can prove that for an arbitrary circle $x_1^2 + x_2^2 = R^2$, where R is so large that κ lies completely inside the circles $x_1^2 + x_2^2 < R^2$, we have the equation

$$\int_{x_1^2 + x_2^2 = R^2} \frac{\partial U_1}{\partial n}\, d\Sigma = 0 \tag{4.9}$$

Taking the limit in the last equation as $R \to \infty$, we obtain that $B_1 = 0$.

The asymptotic form of the solution of the Neumann problem as $R = \sqrt{(X_1^2 + X_2^2)} \to \infty$.

$$\Delta V_j = 0, \quad \partial V_j / \partial n \Big|_{\partial\kappa} = -\partial X_j / \partial n \Big|_{\partial\kappa}, \quad V_j \underset{R\to\infty}{=} O(1/R) \tag{4.10}$$

obviously has the form

$$V_j = d_{jh} \partial \ln R / \partial X_h + O(1/R^2) \tag{4.11}$$

where d_{jh} are certain coefficients. Formula (4.8) now gives

$$U_1 = D_i d_{ih} \partial \ln R / \partial X_h + O(1/R^2) \tag{4.12}$$

Comparing (4.12) with expansion (4.1) in the ring (4.3), we obtain

$$-\frac{1}{2\pi} B_{2j} \partial \ln r / \partial x_j \sim D_i d_{ih} \partial \ln R / \partial X_h, \quad B_{2j} = -2\pi D_i d_{ij} \tag{4.13}$$

The matrix $\| d_{ij} \|$ is an interesting global characteristic of the region κ, which we will discuss below.
We will now find the coefficient B_2 and function U_2 (see 4.1) and (4.2)). For U_2 we again have Laplace's equation. Equation (4.6) and expansion (4.7) lead to the boundary condition

$$\partial U_2 / \partial n \big|_{\partial\kappa} = -D_{ij} X_i \partial X_j / \partial n \tag{4.14}$$

The behaviour of U_2 at infinity can be found from the asymptotic equivalence of expansions (4.1) and (4.2) in the ring (4.3). We obtain

$$U_2 = (2\pi)^{-1} B_2 \ln r + O(1/R^2); \quad R = \sqrt{X_1^2 + X_2^2}, \quad r = R\varepsilon \tag{4.15}$$

Note that, in view of the fact that U_2 is harmonic, we have

$$\int_{\partial\kappa} \frac{\partial U_2}{\partial n} ds = \int_{\Sigma_R} \frac{\partial U_2}{\partial n} d\Sigma_R$$

Taking the limit here as $R \to \infty$ and using relation (4.15) and (4.16) we obtain

$$-\int_{\partial\kappa} D_{ij} X_i \frac{\partial X_j}{\partial n} ds = B_2$$

Using Green's integral formula, we can convert this relation to the form

$$B_2 = -\int_{\kappa} dX_1 dX_2 \delta_{ij} D_{ij} = -(D_{11} + D_{22}) \operatorname{mes}\kappa \tag{4.16}$$

where mes κ is the area of the region κ. The formulae $B_1 = 0$, (4.13) and (4.16) give the required expressions for the leading terms of the expansion (4.1).

The quantities d_{ij} (see (4.13)) form a tensor, which was considered in [6] in connection with other problems. We know that the matrix of this tensor $\| d_{ij} \|$ is symmetric and positive-definite. For a similar expansion (or contraction) of the region κ with a coefficient of proportionality ε, the components of the tensor d_{ij} are multiplied by ε^2, which enables us to "remove" the parameter ε from the final formula for g_r

$$g_r \cong [-(D_{11} + D_{22}) \operatorname{mes}\kappa_\varepsilon g_0(\omega, O) - 2\pi D_i d_{ij}(\kappa_\varepsilon) \partial g_0 / \partial x_i^0] + \ldots \tag{4.17}$$

$$D_i = \partial g_0(\omega_0, O) / \partial x_i^0, \quad D_{ii} = \partial^2 g_0(\omega_0, O) / \partial x_i^{02}$$

(the differentiation is carried out with respect to the coordinates of the point O).

Expression (4.17) is symmetrical about the points ω_0 and ω by virtue of the symmetry of the matrix $\| d_{ij}(\kappa_\varepsilon) \|$ and the set equations

$$D_{11} + D_{22} = \Delta_S g_0 = (\tau^2 + 1/4)\, g_0(\omega, O)$$

The symmetry of the principal part of the outer expansion of g_r might have been expected since the function g_r is symmetric with respect to ω_0 and ω.

5. AN EXPRESSION FOR THE RADIATION PATTERN

Expression (4.1), where g_r has the form (4.17), gives the required formula

$$f(\omega,\omega_0) \cong -\frac{i}{\pi}\int_{-\infty}^{+\infty} e^{\pi\tau}\left[-(D_{11}+D_{22})\,\mathrm{mes}\,\kappa_\varepsilon g_0 - 2\pi D_i d_{ij}\frac{\partial g_0}{\partial x_j^0}\right]\tau\, d\tau \tag{5.1}$$

$$g_0(\omega,O) = -\frac{1}{4\,\mathrm{ch}\,\pi\tau} P_{-\frac{1}{2}+i\tau}(-\cos\theta), \quad \theta = \mathrm{dist}(\omega,O) \tag{5.2}$$

Here D_i and D_{ii} have the form (4.20).

The integrand depends in a quite complex way on τ, but, using the equation

$$\int_{-\infty}^{+\infty}\frac{\mathrm{sh}\,\pi\tau\, P_{-\frac{1}{2}+i\tau}(-\cos\theta)\, P_{-\frac{1}{2}+i\tau}(-\cos\theta')\,\pi\tau\, d\tau}{\mathrm{ch}^2\,\pi\tau} = -\frac{2}{\cos\theta+\cos\theta'} \tag{5.3}$$

which holds when $\theta + \theta' > \pi$, $\theta \geqslant 0$, $\theta' \geqslant 0$, all the integrals in (5.1) can be evaluated explicitly [7].

We obtain

$$f(\omega,\omega_0) \cong -\frac{i}{4\pi^2}\,\mathrm{mes}\,\kappa_\varepsilon \frac{1+\cos\theta\cos\theta'}{(\cos\theta+\cos\theta')^3} - \frac{i}{2\pi} d_{ij}(\kappa_\varepsilon)\, l_i l_j' \frac{\sin\theta\sin\theta'}{(\cos\theta+\cos\theta')^3} \tag{5.4}$$

$\theta = \mathrm{dist}(\omega, O)$, $\theta' = \mathrm{dist}(\omega_0, O)$ l_1, l_2 (respectively l_1', l_2') are the components in the x_1, x_2 system of coordinates of the unit vector which is tangential, at the point O, to the arc of the great circle ωO (respectively $\omega_0 O$). It is assumed that on the arc ωO (respectively $\omega_0 O$) the direction is chosen to be from ω to O (respectively from ω_0 to O). Formula (5.4) is the main result of this paper.

REFERENCES

1. FELSEN L. B., Back scattering from wide-angle at narrow-angle cones. *J. Appl. Phys.* **26**, 2, 138–151, 1955.
2. FELSEN L. B., Plane-wave scattering by small-angles cones. *IRE Trans. Ser. Antennas Propagat.* **5**, 1, 121–129, 1957.
3. SMYSHLYAEV V. P., Diffraction by conical surfaces at high frequencies. *Wave Motion* **12**, 4, 329–339, 1990.
4. BOROVIKOV V. A., *Diffraction by Polygons and Polydedra*. Nauka, Moscow, 1966.
5. IL'IN A. M., *Matching of the Asymptotic Expansions of the Solutions of Boundary-value Problems.* Nauka, Moscow, 1989.
6. POLYA H. AND SZEGÖ G., *Isoperimetric Inequalities in Mathematical Physics.* Nauka, Moscow, 1962.
7. FELSEN L. B., Some definite integrals involving conical functions. *J. Math. and Phys.* **35**, 2, 177–178, 1956.

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